Overlap Dirac Operator, Eigenvalues and Random Matrix Theory

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The properties of the spectrum of the overlap Dirac operator and their relation to random matrix theory are studied. In particular, the predictions from chiral random matrix theory in topologically non-trivial gauge field sectors are tested.

An important property of massless QCD is the spontaneous breaking of chiral symmetry. The associated Goldstone pions dominate the low-energy, finite-volume scaling behavior of the Dirac operator spectrum in the microscopic regime, $1/\Lambda_{QCD} << L << 1/m_\pi$, with L the length of the system [1]. This behavior can be characterized by chiral random matrix theory (RMT). The RMT description of the low-energy, finite-volume scaling behavior is specified by symmetry properties of the Dirac operator and the topological charge sector being considered [2,3]. The RMT predictions are universal in the sense that the symmetry properties, but not the form of the potential matters [4]. Furthermore, the properties can be derived directly from the effective, finite-volume partition functions of QCD of Leutwyler and Smilga, without the detour through RMT [3], though RMT nicely and succinctly describes and classifies all these properties. The topological charge enters the RMT prediction via the number of fermionic zero modes, related to the topological charge through the index theorem. The symmetry properties of the Dirac operator fall into three classes, corresponding to the chiral orthogonal, unitary, and symplectic ensembles [3]. Examples are, respectively, fermions in the fundamental representation of gauge group SU(2), fermions in the fundamental representation of gauge group SU(N_c) with $N_c \geq 3$, and fermions in the adjoint representation of gauge group SU(N_c).

The classification according to the three RMT ensembles is connected to the chiral properties of the fermions [3]. A good non-perturbative regularization of QCD should therefore retain those chiral properties. Until recently such a regularization was not known. The next best thing were staggered fermions, which at least retained a reduced chiral-like symmetry on the lattice. Indeed, staggered fermions were used to verify predictions of chiral RMT, albeit with two important shortcomings: (i) staggered fermions in the fundamental representation of SU(2) have the symmetry properties of the symplectic ensemble, not the orthogonal ensemble as continuum fermions, while adjoint staggered fermions belong to the orthogonal ensemble, not the symplectic one. (ii) staggered fermions do not have exact zero modes at finite lattice spacing, even for topologically non-trivial gauge field backgrounds, and thus seem to probe only the $\nu = 0$ predictions of chiral RMT.

The development of the overlap formalism for chiral fermions on the lattice [5] led to the massless overlap Dirac operator, a lattice regularization for vector-like gauge theories that retains the chiral properties of continuum fermions on the lattice [6]. In particular, the continuum predictions of chiral RMT should apply. Overlap fermions have exact zero modes in topologically non-trivial gauge field backgrounds [7], allowing, for the first time, verification of the RMT predictions in $\nu \neq 0$ sectors. The nice agreement we

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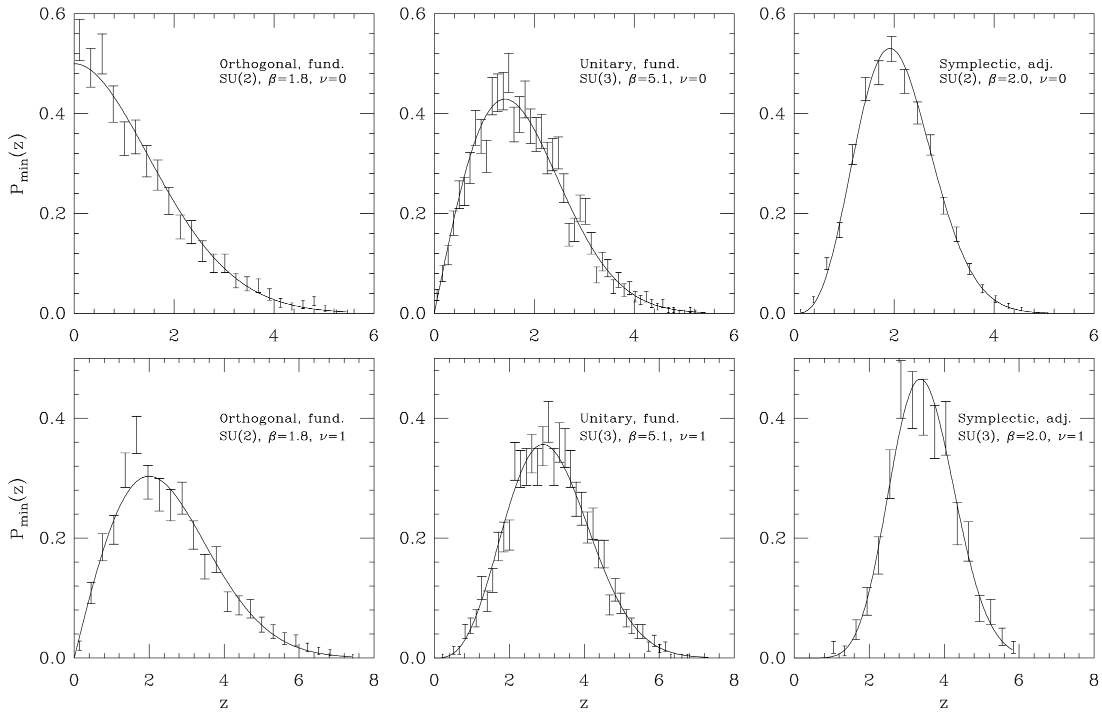

Figure 1. Plots of $P_{\min}(z)$ versus z for the various ensembles in the lowest two topological sectors. The curve in each plot is a fit to the prediction from random matrix theory with the best value for the chiral condensate.

shall describe further validates the chiral RMT predictions and strengthens the case for the usefulness of the Overlap regularization of massless fermions.

The massless overlap Dirac operator [6] is given by

$$D = \frac{1}{2}\left[1 + \gamma_5 \epsilon(H_w(m))\right] \ . \qquad (1)$$

Here, $\gamma_5 H_w(-m)$ is the usual Wilson-Dirac operator and ϵ denotes the sign function. The mass m has to be chosen to be positive and well above the critical mass for Wilson fermions but below the mass where the doublers become light. We are interested in the low lying eigenvalues of the hermitian operator $H = \gamma_5 D$ described in Ref. [7]. We will use the Ritz algorithm [8] applied to H^2 to obtain the lowest few eigenvalues. The numerical algorithm involves the action of H on a vector and for this purpose one will have to use a representation of $\epsilon(H_w(m))$. We used the rational approximation discussed in Ref. [7].

We computed the distribution of the lowest lying eigenvalue of the overlap Dirac operator in the fundamental representation on pure gauge SU(2) configurations with $\beta = 1.8$ as an example of the chiral orthogonal ensemble, on pure gauge SU(3) configurations with $\beta = 5.1$ as an example of the chiral unitary ensemble, and in the adjoint representation on pure gauge SU(2) configurations with $\beta = 2.0$ as an example of the chiral symplectic ensemble. The lattice size was 4^4 in all cases. Chiral RMT predicts that these distributions are universal when they are classified according to the three ensembles and according to the number of exact zero modes ν within each ensemble and then considered as functions of the rescaled variable $z = \Sigma V \lambda_{\min}$. Here V is the volume and Σ is the infinite volume value of the chiral condensate $\langle\bar{\psi}\psi\rangle$ determined up to an overall wave function normalization, which is dependent in part on the Wilson–Dirac mass m. RMT gives the distribution of the rescaled lowest eigenvalue.

A collection of the necessary formulae for the distribution of the lowest eigenvalue, $P_{\min}(z)$, can be found in [9].

We compare the RMT predictions with our data in Fig. 1. If Σ is known, the RMT predictions for $P_{\min}(z)$ are parameter free. On the rather small systems that we considered here, we did not obtain direct estimates of Σ. Instead, we made one-parameter fits of the measured distributions, obtained from histograms with jackknife errors, to the RMT predictions, with Σ the free parameter. Our results and some additional information are given in Table 1. We note the consistency of the values for Σ obtained in the $\nu = 0$ and $\nu = 1$ sectors of each ensemble. Alternatively, we could have used the value of Σ obtained in the $\nu = 0$ sector, to obtain a parameter free prediction for the distribution of the rescaled lowest eigenvalue in the $\nu = 1$ sector. Obviously, the predictions would have come out very well.

With the fermions in the fundamental representation, we found 81 (for SU(2)), and 147 (for SU(3)) configurations with two zero modes and 1 and 3 with three zero modes. For the orthogonal ensemble, we are not aware of a prediction for $P_{\min}(z)$ in the $\nu = 2$ sector, while for the unitary ensemble our data, albeit with very limited statistics, agrees reasonably well with the parameter free prediction with Σ from Table 1.

For fermions in the adjoint representation, we keep only one of each pair of degenerate eigenvalues so $\nu = 1$ is the sector with two exact zero modes. Such configurations cannot be assigned an integer topological charge since integer charges give rise to zero modes in multiples of four [10], and we note there are a significant number of configurations with two zero modes as seen in Table 1. The good agreement with the RMT prediction found in this case lends further support to the existence of configurations with fractional topological charge [10].

We have tested the predictions of chiral random matrix theory using the overlap Dirac operator on pure gauge field ensembles. We find the distribution of the lowest eigenvalue in the different topological sectors fits well with the predictions of chiral RMT, with compatible values for the chiral condensate from the different topological sectors.

Table 1
The chiral condensate, Σ, from fits of the distribution of the lowest eigenvalue to the RMT predictions. The third column gives the Wilson-Dirac mass parameter used, the fourth the number of configurations, N_ν, in each topological sector.

Repr.	β	m	ν	N_ν	Σ
SU(2) fund.	1.8	2.3	0	1293	0.2181(51)
SU(2) fund.	1.8	2.3	1	1125	0.2155(37)
SU(3) fund.	5.1	2.0	0	2714	0.1655(16)
SU(3) fund.	5.1	2.0	1	2136	0.1660(12)
SU(2) adj.	2.0	2.3	0	1251	0.2900(30)
SU(2) adj.	2.0	2.3	1	254	0.2931(45)

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